AMERICAN JOURNAL OF SOCIAL SCIENCE
ISSN: 2996-5306 (online) | ResearchBib (IF) = 10.67 IMPACT FACTOR
Volume-3| Issue-7| 2025 Published: |30-07-2025|

ANALYSIS OF PENDULUM MOTION USING DIFFERENTIAL EQUATIONS AND

https://doi.org/10.5281/zenodo.16959695

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Annotation

In this article, the oscillatory motion of physical and mathematical pendulums is studied based on mathematical modeling. Initially, the physical structure of each pendulum, the forces causing their oscillations, and their mechanical properties are analyzed, and the motion process is expressed in the form of differential equations. Newton's second law and fundamental physical principles are employed in deriving the motion equations. The analyses conducted in the article aim to highlight the theoretical and practical significance of differential equations derived from physical models. Through this, a deeper understanding of pendulum motion is achieved, along with possibilities for applying it to real physical systems and using it as an educational tool in modern learning processes. The theoretical and practical results obtained in the article illustrate the role of differential equations in modeling physical problems and justify the effectiveness of using mathematical software packages in solving them. Moreover, these methods and results are important for their effective use in teaching mechanics courses and organizing laboratory work in higher education.

Keywords

differential equation, mathematical pendulum, physical pendulum, general solution, angular acceleration, mass, moment of inertia.

Introduction

The process of studying mechanical motion is usually reduced to formulating and solving differential equations. However, the solutions of these equations are not always expressed in terms of elementary functions. Therefore, mathematical software packages can be effectively used for analyzing solutions. Below, we will derive the pendulum’s equation of motion and study it using the Maple software package.

Problem Statement and Solution

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The motion of a pendulum is used from both theoretical and practical perspectives to solve technological and scientific problems.

A mathematical pendulum is a system consisting of a massless, inextensible rod of length *l*, suspended at point *O*, with a point mass *m* attached to it. This pendulum moves in a vertical plane passing through point *O*. The position of the pendulum is determined by the angle φ between the vertical axis directed downward and the rod, where $\varphi = \varphi(t)$ ($t-$ time). According to Newton's second law, a force acts on point *M* with mass *m*

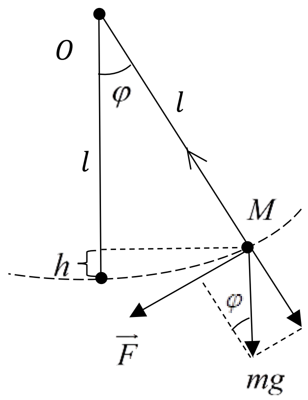

Figure 1.

$\vec{F} = m\vec{a}$ ($\vec{a}-$ acceleration vector, F the sum of the tension force and the gravitational force). This vector equation is projected onto the tangent to the trajectory along which point M moves. The projection of the tension force is equal to zero, while the projection of the gravitational force is equal to $F = mg\sin\varphi$ (sm. Figure 1). The projection of the acceleration vector is equal to $a = l\varphi''$ (where φ'' - angular acceleration). Thus,

$$ml\varphi'' = -mg\sin\varphi. \qquad (1)$$

This equation is an autonomous second-order differential equation with respect to the unknown function $\varphi = \varphi(t)$. Its order can be reduced by replacing the variable t with φ, which is possible using the equation (1):

$$\varphi'' = \frac{d\varphi'}{dt} = \frac{d\varphi'}{d\varphi} \cdot \frac{d\varphi}{dt} = \frac{d}{d\varphi}\left(\frac{\varphi'^2}{2}\right),$$

from which the equation follows:

$$\frac{d}{d\varphi}\left(\frac{\varphi'^2}{2}\right) = -\frac{g}{l}\sin\varphi \text{ or } d\left(\frac{\varphi'^2}{2}\right) = -\frac{g}{l}\sin\varphi\, d\varphi \qquad (2)$$

By integrating equation (2), we obtain:

$$\frac{\varphi'^2}{2} = \frac{g}{l}\cos\varphi + C_1 \ (C_1 = const).$$

Let's substitute into this equation $\omega^2 = \frac{g}{l}$, $C_1 = -\frac{g}{l} + C$ ($C = const$) and we get:

$$\frac{\varphi'^2}{2} + \omega^2(1 - \cos\varphi) = C. \qquad (3)$$

Equation (3) expresses the conservation of total energy of the system with mass *m*, since the kinetic energy is equal to

$$\frac{mv^2}{2} = \frac{m(l\varphi')^2}{2} = ml^2 \cdot \frac{\varphi'^2}{2},$$

and the potential energy is

$mgh = mgl(1-\cos\varphi)$.

thus, the total energy of the system:

$$E = \frac{mv^2}{2} + mgh = ml^2 \cdot \frac{\varphi'^2}{2} + mgl(1-\cos\varphi) = ml^2\left(\frac{\varphi'^2}{2} + \frac{g}{l}\cdot(1-\cos\varphi)\right).$$

Therefore, during the motion:

$$\frac{\varphi'^2}{2} + u(\varphi) = C, u(\varphi) = \omega^2(1-\cos\varphi)\ (C = const).$$

From here:

$$\varphi' = \pm\sqrt{2(C - u(\varphi))}. \quad (4)$$

This equation in the phase space (φ, φ') defines the phase trajectories.

If $C < 0$, then the trajectory does not exist. If $C = 0$, then $\varphi = \pi k, k \in \mathbb{Z}$ these are the equilibrium points of the system.

If $0 < C < 2\omega^2$, then the trajectories will represent closed curves corresponding to the periodic oscillations of the system.

If $C > 2\omega^2$, then the phase trajectories will be open curves, corresponding to rotational motion around point O.

In the general case, the solutions of equation (4) cannot be expressed in terms of elementary functions. The solutions can be found using Jacobi elliptic functions.

If, in the solution, the angle φ varies over a small interval, then $\sin\varphi \approx \varphi$, and (1) the equation will take the form of a harmonic oscillator:

$\varphi'' + \omega^2\varphi = 0$.

The solutions of this equation are expressed in terms of elementary functions and have the form:

$\varphi = A\cos(\omega t + \varphi_0)$,

where $A -$ amplitude, $\varphi_0 -$ initial phase. This solution describes harmonic oscillations with a period of $T = \frac{2\pi}{\omega}$.

Therefore, small oscillations of a mathematical pendulum represent harmonic oscillations.

Let's study the trajectories of the motion of a mathematical pendulum in phase space for $\omega = 1$ using the Maple package.

```
> ω := 1
ω := 1
```

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> $with(plots) : p1 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 1, \phi=-4..4, \phi'=-2..2, scaling = constrained, thickness = 2, color = \text{"Niagara DarkOrchid"}\right)$

$p1 := PLOT(...)$

> $p2 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 0.4, \phi=-4..4, \phi'=-2.7..2.7, scaling = constrained, thickness = 2, color = blue\right)$

$p2 := PLOT(...)$

> $p3 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 1.98, \phi=-4..4, \phi'=-2.5..2.5, scaling = constrained, thickness = 2, color = green\right)$

$p3 := PLOT(...)$

> $p4 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 1.73, \phi=-4..4, \phi'=-2.5..2.5, scaling = constrained, thickness = 2, color = \text{"Generic Blue Purple"}\right)$

$p4 := PLOT(...)$

> $p5 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 3.2, \phi=-4..4, \phi'=-2.7..2.7, scaling = constrained, thickness = 2, color = black\right)$

$p5 := PLOT(...)$

> $p6 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 2.5, \phi=-4..4, \phi'=-2.5..2.5, scaling = constrained, thickness = 2, color = black\right)$

$p6 := PLOT(...)$

> $p7 := implicitplot\left(\frac{(\phi')^2}{2} + \omega\cdot(1-\cos(\phi)) = 4.1, \phi=-4..4, \phi'=-3.2..3.2, scaling = constrained, thickness = 2, color = black\right)$

$p7 := PLOT(...)$

> $display(p1, p2, p3, p4, p5, p6, p7)$

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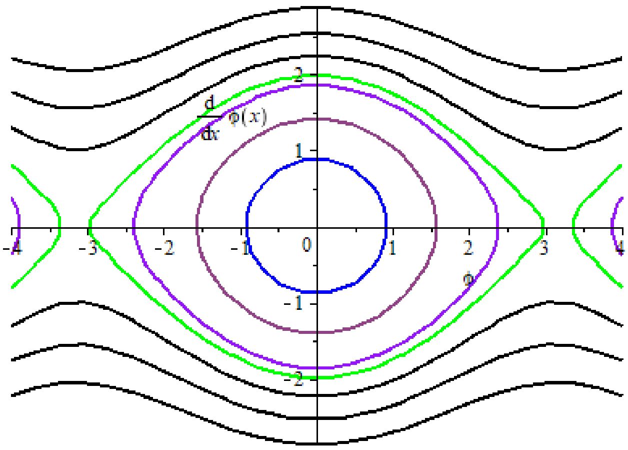

Let's solve the last equation

$$\varphi' = \pm\sqrt{2\left(c_1 - u(\varphi)\right)}$$

with the initial condition $\varphi(0)=\varphi_0,\ \varphi'(0)=0$.

The solution is expressed in terms of elliptic functions.

We will study this solution using the Maple package. The solution is expressed in terms of an integral.:

$$\pm\int_{\varphi_0}^{\varphi}\frac{da}{\sqrt{2\left(u(\varphi_0)-u(a)\right)}}=t.$$

In this case, the + (plus) sign is used when the sign φ' is positive. When the motion of the pendulum starts at an angle φ_0 without initial velocity, we find its period T.

The period T is derived from the following equation

$$-\int_{\varphi_0}^{0}\frac{da}{\sqrt{2\left(u(\varphi_0)-u(a)\right)}}=\frac{T}{4} \text{ or } \int_{0}^{\varphi_0}\frac{da}{\sqrt{2\left(u(\varphi_0)-u(a)\right)}}=\frac{T}{4}.$$

We will calculate this period in the Maple software package when $\varphi_0=\frac{\pi}{3}$.

$$> int\left(\frac{4}{\sqrt{2\cdot\left(\cos(\phi)-\cos\left(\frac{\pi}{3}\right)\right)}}, \phi=0..\frac{\pi}{3}, numeric\right)$$

6.743001419

Thus, in the mathematical pendulum, the period of oscillatory motion that starts at an angle $\frac{\pi}{3}$ under its own weight is equal to $T=6,743001419$.

For a mathematical pendulum in the equation with small angular deviations, the oscillation period is equal to:

$$T = \frac{2\pi}{\omega}.$$

If $\omega = 1$, then the period will be equal to 2π. It is evident that its period is greater 2π in large motions.

In a similar way, the equation of motion of a physical pendulum can be studied. A physical pendulum is a rigid body that oscillates freely under the action of gravity about an arbitrary fixed horizontal axis that does not pass through its center of mass.

The equation of motion of a physical pendulum of mass m, rotated from the equilibrium position by an angle φ, is reduced to a differential equation of the form $\frac{d^2\varphi}{dt^2} + \omega^2 \sin\varphi = 0$, where $\omega^2 = \frac{mgl}{I}$, I – moment of inertia, l – the distance from the center of mass of the body to the axis of rotation, g – acceleration due to gravity, m – mass of the body. The solutions of this differential equation were studied above.

Conclusion

The equation of small oscillations of a mathematical pendulum is similar to the equation of harmonic oscillations. This equation can be used to determine the oscillation period, angular frequency, and other dynamic properties of the pendulum. It follows that the period of small oscillations does not depend on the initial conditions. However, in the general case (for large oscillations), the oscillation period depends on the initial conditions.

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