SOME PROBLEMS OF SPHERICAL TRIGONOMETRY

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Abstract

This article discusses some theoretical problems of spherical trigonometry. The article provides the definition of a spherical angle, one of the fundamental concepts of spherical geometry. A theorem stating that a spherical angle is measured by the arc of a great circle is proved. In addition, important geometric results related to a great circle and its pole are presented. The paper also considers the concept of a spherical zone, its bases, and its height. Problems concerning the calculation of the lateral surface area of a spherical segment and a spherical zone are examined. The surface area generated by rotating a segment or a radius about an axis is analyzed. It is substantiated that the surface area of a sphere can be expressed as the product of its diameter and the circumference of a great circle. The theorems presented in the article serve as an important theoretical basis for studying spherical trigonometry. These results can be applied in spherical geometry, astronomy, navigation, and the solution of practical problems.

Keywords

spherical trigonometry, spherical geometry, spherical angle, great circle, spherical zone, spherical segment, surface area of a sphere, spherical triangle.

Introduction. Spherical trigonometry is one of the important branches of mathematics that studies the relationships between the sides and angles of geometric figures located on the surface of a sphere, in particular spherical triangles. While in plane trigonometry triangles are considered in the Euclidean plane, in spherical trigonometry triangles are formed on the surface of a sphere by arcs of great circles. Therefore, this field differs from ordinary plane geometry and has its own specific features, such as the sum of the angles being greater than 180 degrees, the sides being expressed by arcs, and the trigonometric formulas having special forms.

The theoretical and practical significance of spherical trigonometry is very broad. Since ancient times, it has been applied in such fields as astronomy, geodesy, cartography, marine and air navigation, and the determination of the coordinates of celestial bodies. In particular, problems such as representing the Earth as a body close to a sphere, determining the positions of stars and planets on the celestial sphere, and calculating the shortest distance between two points are closely related to the methods of spherical trigonometry. Even today, this field has not lost its relevance and is widely used in satellite navigation, geographic information systems, oceanography, space research, and numerical algorithms.

Among the existing scientific and methodological literature in the Uzbek language, the textbook *Spherical Trigonometry* by Q. Mamasoliyev and B. Mardonov [1] is of particular importance. This textbook is devoted directly to the subject of spherical trigonometry and systematically presents spherical triangles, their basic elements, trigonometric relationships between sides and angles, fundamental formulas, and methods for solving problems. In this regard, this source is considered an important educational resource for the theoretical study of spherical trigonometry and for solving practical problems.

M.Kh. Egamov's methodological manual entitled *Technology for Project-Based Teaching of Spherical Trigonometry* [2] addresses the topic from the perspective of teaching methodology. This source examines issues related to the effective organization of spherical trigonometry in the educational process, the step-by-step design of topics, and the development of students' spatial imagination and logical thinking. This demonstrates the methodological significance of spherical trigonometry not only as a theoretical discipline, but also within the educational process.

Spherical trigonometry is also closely related to astronomy. The section "Fundamentals of Spherical Astronomy" in M. Mamadazimov's textbook *General Astronomy* plays an important role in understanding this relationship [3]. In determining the coordinates of celestial bodies on the celestial sphere, describing their apparent motion, and explaining such concepts as the horizon, meridian, and ecliptic, formulas of spherical trigonometry and the theory of spherical triangles are used. Therefore, knowledge of the fundamentals of spherical trigonometry is essential for a deeper understanding of astronomical problems.

In recent years, interest in spherical geometry and spherical trigonometry has also been increasing in foreign scientific and educational literature. In the educational manual *Spherical Geometry for School Students* by V.A. Smirnov and I.M. Smirnova [4], the basic concepts, properties, and theorems of spherical geometry are presented in a simple and systematic manner. This manual is a useful source for developing the geometric foundation required for the study of spherical trigonometry. In the educational manual *Spherical Astronomy* by I.F. Bikmaev and

V.V. Shimansky [5], the fact that the first chapter is directly devoted to the "elements of spherical trigonometry" indicates that this field forms the theoretical basis of spherical astronomy.

Glen Van Brummelen's monograph *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry* occupies an important place in elucidating the historical development and scientific significance of spherical trigonometry [6]. It extensively discusses the role of spherical trigonometry in ancient astronomy, mathematics of the Islamic world, European science, and the history of navigation. This work shows that spherical trigonometry is not merely a collection of formulas, but an important mathematical achievement that arose from humanity's need to study celestial bodies and determine directions in space.

The article "Remembering Spherical Trigonometry" by John Conway and Alex Ryba [7] is significant for its concise, comprehensible, and modern methodological presentation of the topic. This article proposes convenient approaches for memorizing the basic formulas of spherical trigonometry, understanding their meaning, and applying them. M.A. Whittlesey's work *Spherical Geometry and Its Applications* explains spherical geometry and spherical trigonometry through a modern mathematical presentation and demonstrates their applicability in various practical fields [8].

The development of modern science and technology is revealing new areas of application for spherical trigonometry. In the section "Spherical Trigonometry and Distance Computation" written by Chunyan Li [9], the application of spherical trigonometry to distance computation problems in oceanography and time series data analysis is discussed. In Jean-Marc Delosme's article "A Signal Flow Graph Approach to the Resolution of Spherical Triangles Using CORDIC" [10], a modern algorithmic method for solving spherical triangles is proposed. This study shows that spherical trigonometry is developing in connection with numerical computation methods, signal flow graphs, and CORDIC algorithms. A.A. Olsen's chapter "Spherical Trigonometry" explains spherical trigonometry in relation to celestial navigation and reveals its practical navigational significance [11].

The analysis of the above-mentioned literature shows that spherical trigonometry is a branch of mathematics that is theoretically rich and of great practical importance. It is closely connected with spherical geometry, astronomy, navigation, geodesy, cartography, distance computation, and modern computational algorithms. Moreover, in the process of teaching this topic, there are opportunities to broaden students' spatial imagination, develop their trigonometric thinking, and form skills in the mathematical modelling of real-life problems.

From this point of view, an in-depth study of the topic "Spherical Trigonometry" is relevant for theoretical mathematics, applied sciences, and teaching methodology. This article analyzes the fundamental concepts of spherical trigonometry, its historical development, its treatment in scientific and methodological sources, and its modern areas of application. This serves to reveal more comprehensively the contemporary scientific and practical significance of spherical trigonometry.

MAIN RESULTS

Definition 1. The intersection of two arcs of great circles is called a **spherical angle**. A spherical angle is considered to be equal to the angle formed by the tangents drawn to these arcs at their point of intersection.

Theorem 1. A spherical angle is measured by the arc of the great circle lying between its sides. The pole of this great circle is the vertex of the given angle.

Proof. Suppose that the arcs PA and PB intersect at the point P, and that they are arcs of great circles. The straight lines PA' and PB' are tangent to the sphere at the point P along these arcs. Let the arc AB be considered as an arc of the great circle drawn with P as its pole, and let it be the part lying between the arcs PA and PB. We prove that the angle $\angle APB$ is measured by the arc AB.

In the plane POB, the segment PB' is perpendicular to PO. Moreover, $OB \perp PO$ and $PB' \parallel OB$. Similarly, $PA' \parallel OA$. Hence, $\angle A'PB' = \angle AOB$.

However, the angle $\angle AOB$ is measured by the arc AB. Therefore, according to Definition 1, the angle $\angle APB$ is also measured by the arc AB.

Corollary 1. All arcs of great circles passing through the pole of a given great circle are perpendicular to the given great circle.

Corollary 2. A spherical angle has the same measure as the dihedral angle between the planes of the great circles forming this angle.

Definition 2. The part of a sphere lying between two planes is called a **spherical zone**. The circles formed on the sphere by these planes are called the bases of the zone. The distance between the planes is called the height of the zone. If one of the planes is tangent to the sphere and the other intersects it, then such a zone is called a one-base zone, or a **spherical segment**. If both planes are tangent to the sphere, then the zone determines the entire sphere.The lateral surface area of a spherical zone is calculated by the formula

$$S = 2\pi RH,$$

where R is the radius of the sphere and H is the height of the zone. This formula will be derived below.

If a great circle is rotated about its diameter, then any of its arcs generates a spherical zone. In the figure, when the great circle $ABCD$ is rotated about the diameter PP', the arc AC generates a spherical zone.

Theorem 2. When the radius rrr of a sphere is rotated about a certain axis, the area of the surface generated is equal to the product of the length of the projection of this radius onto the axis and the circumference of the circle whose radius is equal to the length of the perpendicular drawn from the axis to the midpoint of the segment.

Proof. Let the axis Ox and the segment AB be given. Let M be the midpoint of the segment AB, and let CD be the projection of the segment AB onto the axis Ox. Let MO be perpendicular to Ox, and let MR also be perpendicular to AB. Denote by S the area of the surface generated by rotating the segment AB about the axis Ox. We prove that $S = CD \cdot 2\pi MR$.

1. Consider the case when the segment AB is parallel to the axis Ox. In this case, $CD = AB$, $MR = MO$, and S is the lateral surface area of a right circular cylinder: $S = MO \cdot 2\pi \cdot MO$.

2. Consider the case when the segment AB is not parallel to the axis Ox and does not intersect it. In this case, the surface S is the lateral surface area of a truncated cone. Draw AE parallel to the axis Ox. The triangle ΔAEB is similar to the triangle ΔMOR. Hence, $MO : AE = MR : AB$, or $AB \cdot MO = AE \cdot MR$.

According to Theorem 1, the lateral surface area of the truncated cone is equal to $S = CD \cdot 2\pi \cdot MR$.

This equality can also be verified by comparing it with the standard formula for the lateral surface area of a truncated cone: $S = \pi(R + r)l$.

3. If the point A lies on the axis Ox, then $AE = CD = AD$.

According to the theorem, the lateral surface area of the cone is

$S = CD \cdot 2\pi MR$.

Indeed, since ΔABD is similar to ΔMOR, we have $\frac{AB}{MR} = \frac{AD}{MO}$ or $AD \cdot MR = MO \cdot AB$.

Since the segment MO is the midline of the triangle ΔABD, it follows that $S = CD \cdot 2\pi MR$. This coincides with Theorem 1. Theorem 1. is proved.

Theorem 2. The surface area of a sphere is equal to the product of its diameter and the circumference of a great circle: $S = 2\pi Rd$.

Proof. Suppose that a sphere is obtained by rotating the semicircle $ABCDE$ about its diameter AE. Here, S denotes the surface area of the sphere, R is the radius of the sphere, and d is the diameter of the sphere. We prove that $S = 2\pi Rd$.

Let a regular polygon $ABCDE$ be inscribed in the semicircle. Then, from the center O, draw perpendiculars to the chords AB, BC, CD, DE. These perpendiculars bisect the corresponding chords. Denote the length of each of these perpendiculars by *l*. Draw perpendiculars from the points B, C, and D to AE.

Then, according to Theorem 1, the area of the surface generated by rotating AB about the axis AE is $AB = AB' \cdot 2\pi l$.

Similarly, $BC = BC' \cdot 2\pi l$.

Continuing this process for the remaining segments, we obtain that the surface area generated by the polygon $ABCDE$ is $S' = AE \cdot 2\pi l = 2\pi ld$.

If the number of sides of the polygon is assumed to increase without bound, that is, if $l \to R$, then it follows that $S' \to S$.

Hence, $S = 2\pi Rd$. Theorem 1.7 is proved.

Corollary 1. The surface area of a sphere is equal to the area of four great circles: $S = 4\pi R^2$.

Example. If $R = 10$, then $S = 4\pi \cdot 100 = 1256,64$.

Corollary 2. The ratio of the surface areas of two spheres is equal to the ratio of the squares of their radii.

Corollary 3. The lateral surface area of a spherical zone is equal to the product of its height and the circumference of a great circle: $S = 2\pi RH$.

Corollary 4. The lateral surface area of a one-base spherical zone is equivalent to the area of a circle whose radius is equal to the chord of the arc generating this zone.

CONCLUSION

This section presents the main results concerning the spherical angle, the spherical zone, and the surface area of a sphere, which are among the important concepts of spherical geometry. First, it was shown that a spherical angle is formed by the intersection of two arcs of great circles, and that its measure is determined by the angle between the tangents drawn to these arcs at their point of intersection. In addition, the theorem stating that the measure of a spherical angle is determined by the arc of the great circle lying between its sides was proved. This result makes it possible to relate angles on the surface of a sphere to ordinary plane angles.

According to the corollaries derived from this theorem, all great circles passing through the pole of a given great circle are perpendicular to that great circle. Moreover, it was established that a spherical angle has the same measure as the dihedral angle between the planes of the great circles forming it. This demonstrates the necessity of studying the concepts of spherical geometry in close connection with elements of spatial geometry.

The section also discusses the concept of a spherical zone. The part of a sphere lying between two planes is defined as a zone; the circles bounding it are called the bases of the zone, and the distance between the planes is called its height. In particular, the case of a one-base zone, that is, a spherical segment, was also considered. It was shown that the lateral surface area of a spherical zone can be expressed in terms of the radius and height of the zone, and the practical importance of this formula for calculating areas of regions on the surface of a sphere was substantiated.

Furthermore, a theorem concerning the calculation of the surface area generated by rotating a radius or a segment about an axis was proved. This theorem makes it possible to interpret the lateral surface areas of cylinders, truncated cones, and cones within a unified geometric approach. As a result, the interrelation between such concepts as projection, perpendicular distance, and circumference in determining the surface areas of solids of revolution was revealed.

As the final result of the section, it was proved that the surface area of a sphere is equal to the product of its diameter and the circumference of a great circle. Through this result, the classical formula for the surface area of a sphere was derived on the basis of geometric reasoning. By inscribing a regular polygon in a semicircle and rotating its sides about the diameter, it was shown that the sequence of generated surfaces approaches the surface of the sphere. On this basis, the formula for the surface area of a sphere was justified in connection with the concept of a limit.

In general, these results serve as an important theoretical foundation for a deeper understanding of the basic concepts of spherical geometry, for calculating angles and surface areas on the sphere, and for analyzing spherical figures from the perspective of spatial geometry. The definitions and theorems presented form the necessary geometric basis for studying spherical trigonometry and provide a solid foundation for subsequent topics concerning spherical triangles, their elements, and trigonometric relationships.

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