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NUMERICAL SOLUTION OF THE DIRICHLET PROBLEM FOR A STATIONARY NONLINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Bobur Islomov[20]*, Jasurbek Sharipov and Tojiddin Khojiev****

* *King Abdullah University of Science and Technology, Thuwal, Saudi Arabia*

** *National University of Uzbekistan, Tashkent, Uzbekistan*

Emails: *bobur.islomov@kaust.edu.sa , jasurbeksharipov55555@gmail.com and tojiddin542018@gmail.com.*

Abstract

This paper presents a numerical solution to the Dirichlet problem applied to a stationary nonlinear system of partial differential equations. The methodology integrates the pseudo-time stepping method with finite difference schemes, utilizing the backward sweep algorithm (Thomas algorithm) and the method of alternating directions (MAD). Comprehensive numerical experiments validate the proposed framework, demonstrating second-order spatial convergence and robust stability across highly coupled nonlinear domains. Furthermore, the study shows that the judicious selection of initial approximations is critical for accelerating steady-state convergence and mitigating computational overhead in strongly nonlinear scenarios.

Keywords

Dirichlet problem, Pseudo-time stepping, Stationary system, Iterative schemes, Sweep method, Numerical modelling and Finite difference method.

1 Introduction

The Dirichlet problem is one of the fundamental problems associated with systems of partial differential equations and plays a crucial role in the mathematical analysis of physical processes [1]. For instance, in heat conduction, elasticity theory, fluid and gas mechanics, electrostatics and potential theory, problems involving Dirichlet boundary conditions are widely encountered [2].

Driven by the increasing complexity of real-world physical models, the development of robust numerical solvers for nonlinear partial differential equation (PDE) systems subject to Dirichlet boundaries has emerged as a focal point in contemporary computational mathematics. Studies show that numerical methods

[20] **Corresponding author.** E-mail address: bobur.islomov@kaust.edu.sa. Takes full responsibility for the integrity of the data and the overall content of the manuscript.

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provide highly effective tools for analysing stationary nonlinear PDE systems [3, 4]. In particular, finite difference techniques, iterative schemes and algorithms designed to satisfy boundary constraints enable the computation of accurate numerical solutions. These methodologies make it possible to approximate solutions to PDE systems using finite difference formulations and to investigate their stability and convergence properties with precision.

2 Literature Review

In recent years, there has been a growing interest in the study of mathematical models describing processes formulated through differential equations. In this context, the contributions of researchers such as M.A.J. Chaplain, B.H. Gilding, L.A. Peletier, T.D. Taylor, E. Ndefo, B.S. Masson, Hiroshi Fujita, D.H. Sattinger, N.T. Danaev, B. Matkerim and D. Zh. Ahmed-Zaki hold significant importance.

A number of problems governed by nonlinear partial differential equations were examined, including models describing water flow and pressure effects in hydraulic structures [5], as well as heat and stress distribution in greenhouse environments [1, 6]. These models were analysed, and general conclusions were drawn regarding their behaviour and applicability. To investigate the qualitative properties of nonlinear partial differential equations, a set of computational schemes, algorithms and software tools were developed [3], and the numerical solutions of these nonlinear problems were subsequently visualized utilizing Python-based computational environments.

3 Methods

3.1 Problem Statement

Let us consider the problem of solving a system of two nonlinear partial differential equations in two variables under Dirichlet boundary conditions.

$$\begin{cases} f_1(v)\left\{a_{11}\frac{\partial}{\partial x}\left(a_{12}\frac{\partial}{\partial x}u\right)+a_{13}\frac{\partial}{\partial y}\left(a_{14}\frac{\partial}{\partial y}u\right)\right\}=F_1(x,y,u,v) \\ f_2(u)\left\{a_{21}\frac{\partial}{\partial x}\left(a_{22}\frac{\partial}{\partial x}v\right)+a_{23}\frac{\partial}{\partial y}\left(a_{24}\frac{\partial}{\partial y}v\right)\right\}=F_2(x,y,u,v) \end{cases} \quad (1)$$

These equations take into account the mutual interaction between the two functions $u(x,y)$ and $v(x,y)$. Each equation is expressed as a sum of second-order partial derivatives of the unknown functions, indicating that the system possesses an elliptic character. The expressions on the left-hand side represent the parameters that describe the mathematical model or the underlying physical process [1], while the right-hand side contains external sources or reaction terms, denoted by $F_1(x,y,u,v)$ and $F_2(x,y,u,v)$.

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In this system, the functions $f_1(v)$ and $f_2(u)$ show that the coefficients of the equations are nonlinear—that is, they depend on the unknown variables themselves—which further increases the complexity of the system. Such equations typically arise in the modelling of heat transfer, electromagnetic wave propagation and diffusion-type systems [7]. Their key feature is that the diffusion rate of each component depends not only on its own gradient but also on the gradient of the other component. Consequently, the spatial diffusion of component u is explicitly modulated by the gradient of v, and reciprocally.

Systems of this type are widely used to describe various physical, biological and chemical processes observed in real-world applications.

The numerical solution of system (1) is sought in a rectangular domain Ω defined as:

$$\Omega = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\} \quad (2)$$

To maintain the correctness of the PDE system (1), the following boundary conditions of the first kind (Dirichlet conditions) must be provided along the boundaries of the domain using the following functions:

$$\begin{cases} u(a,y) = \varphi_{11}(y) \\ u(b,y) = \varphi_{12}(y) \\ u(x,c) = \varphi_{13}(x) \\ u(x,d) = \varphi_{14}(x) \end{cases}, \quad \begin{cases} v(a,y) = \varphi_{21}(y) \\ v(b,y) = \varphi_{22}(y) \\ v(x,c) = \varphi_{23}(x) \\ v(x,d) = \varphi_{24}(x) \end{cases} \quad (3)$$

3.2 Method for Solving Boundary Value Problems for Second-Order Differential Equations

To solve the given system of equations, we use the sweep method, also known as the Thomas algorithm. This approach is one of the most widely employed techniques for obtaining numerical solutions. It is evident that the problem is completely independent of time (stationary). One of the most effective methods for solving such partial differential equations is to transform them into a non-stationary form [1,2]:

$$\begin{cases} B_1 \frac{\partial u}{\partial t} = L_1 u \equiv f_1(v)\{a_{11} \frac{\partial}{\partial x}(a_{12} \frac{\partial u}{\partial x}) + a_{13} \frac{\partial}{\partial y}(a_{14} \frac{\partial u}{\partial y})\} + \delta_1 F_1(x,y,u,v) \\ B_2 \frac{\partial v}{\partial t} = L_2 v \equiv f_2(u)\{a_{21} \frac{\partial}{\partial x}(a_{22} \frac{\partial v}{\partial x}) + a_{23} \frac{\partial}{\partial y}(a_{24} \frac{\partial v}{\partial y})\} + \delta_2 F_2(x,y,u,v) \end{cases} \quad (4)$$

In the context of the operational formulation of the problem, the operators B_1 and B_2 are assumed to be identity operators, such that $B_1 = B_2 = E$, where E denotes the identity matrix. This assumption simplifies the transient part of the governing equations (4) and is consistent with the standard conservative finite

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difference schemes for nonlinear heat conduction problems. The theoretical justification for this operator selection and its impact on the stability of the iterative Method of Alternating Directions is discussed in detail in the classical literature by Samarskii [8]. To ensure the correctness of the problem, appropriate supplementary conditions must be introduced. These conditions are defined as initial values, leading to the addition of function values at $t = 0$:

$$\begin{cases} E\frac{\partial u}{\partial t} = L_1 u \\ u|_{\Omega} = \varphi_1 \\ u|_{t=t_0} = \psi_1(x,y) \\ \left.\frac{\partial u}{\partial t}\right|_{t\to\infty} \to 0 \end{cases} \to \begin{cases} L_1 u = 0 \\ u|_{\Omega} = \varphi_1 \end{cases} \quad \text{and} \quad \begin{cases} E\frac{\partial v}{\partial t} = L_2 v \\ v|_{\Omega} = \varphi_2 \\ v|_{t=t_0} = \psi_2(x,y) \\ \left.\frac{\partial v}{\partial t}\right|_{t\to\infty} \to 0 \end{cases} \to \begin{cases} L_2 v = 0 \\ v|_{\Omega} = \varphi_2 \end{cases} \quad (4.1\text{-}4.2)$$

The convergence of the system is monitored by the condition where the time derivatives vanish, $\partial u/\partial t \to 0$ and $\partial v/\partial t \to 0$ as $t \to \infty$, signifying that the solution has reached its steady state.

This approach, theoretically supported by Godunov and Ryabenkii [9], ensures that the boundary value problem is effectively transformed into a stable iterative process capable of handling nonlinear source terms and complex boundary conditions. This mathematical formulation illustrates the transition from a non-stationary evolutionary process to a stationary state as $t \to \infty$. The convergence of the time-dependent solution to the solution of the original elliptic problem (1) forms the basis of the iterative establishment procedure used in our numerical simulations. To achieve this, an implicit scheme known as the Method of Alternating Directions (MAD) can be employed [10]. In certain literature, this approach is also referred to as the Method of Fractional Steps. The implementation is carried out by formulating the system into a finite-difference representation within the following implicit scheme. In this method, intermediate values $\tilde{u} = u^{s+1/2}$ and $\tilde{v} = v^{s+1/2}$ are calculated within a single time step. Consequently, the values for $\tilde{u}, \tilde{v}$ and u^{s+1}, v^{s+1} are determined sequentially in two stages:

$$\begin{cases} \tilde{u}_t = \theta L_{xx}\tilde{u} + (1-\theta)L_{yy}u^s \\ u_t^{s+1} = (1-\theta)L_{xx}\tilde{u} + \theta L_{yy}u^{s+1} \end{cases} \quad (4.3)$$

$$\begin{cases} \tilde{v}_t = \theta L_{xx}\tilde{v} + (1-\theta)L_{yy}v^s \\ v_t^{s+1} = (1-\theta)L_{xx}\tilde{v} + \theta L_{yy}v^{s+1} \end{cases} \quad (4.4)$$

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Here, different numerical schemes can be obtained depending on the value of the parameter θ: If $\theta = 1/2$, the formulation represents the Crank-Nicolson scheme. If $\theta = 1$, it results in a fully implicit scheme. If $\theta = 0$, it reduces to a fully explicit scheme. This approach is widely recognized in the literature as the *pseudo-time stepping method* (or the *relaxation method*) [10, 11]. The partial derivatives are approximated using the following finite difference relations and then substituted into the system. For the sake of brevity, these approximations are provided for the function $u(x, y)$, while the function $v(x, y)$ is treated in an entirely analogous manner.

To solve the nonlinear coupled system numerically, the domain $\Omega = [a, b] \times [c, d]$ is discretized using a uniform grid with spatial steps $h = (b - a)/m$, $l = (d - c)/n$ and taking $\delta_1, \delta_2 = 1$ in (4). The time derivative and spatial operators are approximated using finite difference schemes as follows:

$$\frac{\partial u}{\partial t} \approx \frac{u_{ij}^{s+1} - \tilde{u}_{ij}}{\tau}, \quad \frac{\partial}{\partial x}\left(a_{ij}\frac{\partial u}{\partial x}\right) \approx a_{i+1j}\frac{u_{i+1j} - u_{ij}}{h^2} - a_{ij}\frac{u_{ij} - u_{i-1j}}{h^2}$$

To maintain the conservative properties and account for spatial heterogeneity, the coefficients a_i are averaged between adjacent grid points:

$$a_i(x, y, u_i) = \frac{a_{i+1}(x, y, u_i) + a_i(x, y, u_i)}{2}$$

The nonlinear source function F is linearized before the final discretization. Two iterative approaches are considered:

• Picard iteration: $F(x, y, u, v) \approx F(x, y, \overset{s}{u}, \overset{s}{v})$, which utilizes values from the previous iteration level s.

• Newton method: $F(x, y, u, v) \approx F(x, y, \overset{s}{u}, \overset{s}{v}) + \mu\frac{\partial F}{\partial u}(\overset{s+1}{u} - \overset{s}{u})$, providing a higher convergence rate through the Jacobian term.

The resulting tridiagonal systems are solved at each half-step using the Method of Alternating Directions (MAD) combined with the Thomas algorithm.

Accordingly, when converting the PDE system (1) into finite difference form, (4.3) and (4.4) schemes are fully employed and approximations are utilized for corresponding differential operators:

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$$\frac{\tilde{u}_{ij}-u_{ij}^{s}}{\tau}=f_1(v_{ij}^{s})\{\frac{\theta}{2}(a_{11i+1}^{s}+a_{11i}^{s})(a_{12i+1j}^{s}\frac{\tilde{u}_{i+1j}-\tilde{u}_{ij}}{h^2}-a_{12ij}^{s}\frac{\tilde{u}_{ij}-\tilde{u}_{i-1j}}{h^2})$$
$$+\frac{(1-\theta)}{2}(a_{13j+1}^{s}+a_{13j}^{s})(a_{14ij+1}^{s}\frac{u_{ij+1}^{s}-u_{ij}^{s}}{l^2}-a_{14ij}^{s}\frac{u_{ij}^{s}-u_{ij-1}^{s}}{l^2})\}$$
$$-F_1(x_i,y_j,u_{ij}^{s},v_{ij}^{s})-\mu_1F_{1u}{}'(x_i,y_j,u_{ij}^{s},v_{ij}^{s})(\tilde{u}_{ij}-u_{ij}^{s}) \quad (5)$$

$$\frac{u_{ij}^{s+1}-\tilde{u}_{ij}}{\tau}=f_1(v_{ij}^{s})\left\{\frac{(1-\theta)}{2}(\tilde{a}_{11i+1}+\tilde{a}_{11i})(\tilde{a}_{12i+1j}\frac{\tilde{u}_{i+1j}-\tilde{u}_{ij}}{h^2}-\tilde{a}_{12ij}\frac{\tilde{u}_{ij}-\tilde{u}_{i-1j}}{h^2})\right.$$
$$\left.+\frac{\theta}{2}(\tilde{a}_{13ij+1}+\tilde{a}_{13ij})(\tilde{a}_{14ij+1}\frac{u_{ij+1}^{s+1}-u_{ij}^{s+1}}{l^2}-\tilde{a}_{14ij}\frac{u_{ij}^{s+1}-u_{ij-1}^{s+1}}{l^2})\right\}$$
$$-F_1(x_i,y_j,\tilde{u}_{ij},v_{ij}^{s})-\mu_1F_{1u}{}'(x_i,y_j,\tilde{u}_{ij},v_{ij}^{s})(u_{ij}^{s+1}-\tilde{u}_{ij}) \quad (6)$$

An analogous set of discretized equations is formulated for the secondary variable, v. The first equation of the system has been formulated into a finite-difference scheme. We must now transform the second equation into its corresponding finite-difference representation. The operations for the second variable are performed in an identical manner to those of the first:

$$\frac{\tilde{v}_{ij}-v_{ij}^{s}}{\tau}=f_2(u_{ij}^{s})\{\frac{\theta}{2}(a_{21i+1}^{s}+a_{21i}^{s})(a_{22i+1j}^{s}\frac{\tilde{v}_{i+1j}-\tilde{v}_{ij}}{h^2}-a_{22ij}^{s}\frac{\tilde{v}_{ij}-\tilde{v}_{i-1j}}{h^2})$$
$$+\frac{(1-\theta)}{2}(a_{23j+1}^{s}+a_{23j}^{s})(a_{24ij+1}^{s}\frac{v_{ij+1}^{s}-v_{ij}^{s}}{l^2}-a_{24ij}^{s}\frac{v_{ij}^{s}-v_{ij-1}^{s}}{l^2})\}$$
$$-F_2(x_i,y_j,u_{ij}^{s+1},v_{ij}^{s})-\mu_2F_{2v}{}'(x_i,y_j,u_{ij}^{s+1},v_{ij}^{s})(\tilde{v}_{ij}-v_{ij}^{s}) \quad (7)$$

$$\frac{v_{ij}^{s+1}-\tilde{v}_{ij}}{\tau}=f_2(u_{ij}^{s})\left\{\frac{(1-\theta)}{2}(\tilde{a}_{21i+1}+\tilde{a}_{21i})(\tilde{a}_{22i+1j}\frac{\tilde{v}_{i+1j}-\tilde{v}_{ij}}{h^2}-\tilde{a}_{22ij}\frac{\tilde{v}_{ij}-\tilde{v}_{i-1j}}{h^2})\right.$$
$$\left.+\frac{\theta}{2}(\tilde{a}_{23ij+1}+\tilde{a}_{23ij})(\tilde{a}_{24ij+1}\frac{v_{ij+1}^{s+1}-v_{ij}^{s+1}}{l^2}-\tilde{a}_{24ij}\frac{v_{ij}^{s+1}-v_{ij-1}^{s+1}}{l^2})\right\}$$
$$-F_2(x_i,y_j,u_{ij}^{s+1},\tilde{v}_{ij})-\mu_2F_{2v}{}'(x_i,y_j,u_{ij}^{s+1},\tilde{v}_{ij})(v_{ij}^{s+1}-\tilde{v}_{ij}) \quad (8)$$

To obtain the numerical solution, we perform simplifications within the above schemes by grouping the terms according to their indices. For the u function, terms are grouped by $i-1, i, i+1$ indices along the OX axis, and by $j-1, j, j+1$ indices along the OY axis [12]. The v function is treated using an identical sequence, resulting in the following system of equations:

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$$\begin{cases} A_1\tilde{u}_{i-1j} + B_1\tilde{u}_{ij} + C_1\tilde{u}_{i+1j} = D_1 \\ A_2u_{ij-1}^{s+1} + B_2u_{ij}^{s+1} + C_2u_{ij+1}^{s+1} = D_2 \\ A_3\tilde{v}_{i-1j} + B_3\tilde{v}_{ij} + C_3\tilde{v}_{i+1j} = D_3 \\ A_4v_{ij-1}^{s+1} + B_4v_{ij}^{s+1} + C_4v_{ij+1}^{s+1} = D_4 \end{cases} \quad (9)$$

The matrix coefficients A_k, B_k, C_k and the right-hand side vectors D_k are derived by substituting the finite difference approximations (5)-(8) into the generalized algebraic form (9).

The sweep method (Thomas algorithm) is an efficient technique for solving tridiagonal systems and consists of two stages. The Thomas algorithm provides a computationally efficient framework for resolving the resulting tridiagonal systems via a two-stage procedure: a forward elimination phase to determine the intermediate coefficients α and β, followed by a backward substitution phase to compute the final spatial distributions of u and v, sometimes called the reverse sweep in the literature [3].

The coefficients for the backward sweep procedure are defined as follows:

$$\tilde{u}_{ij} = \alpha_{1i}\tilde{u}_{i+1j} + \beta_{1i}, \quad i = n-1, n-2, \dots, 2. \quad (10)$$

$$\alpha_{11} = 0; \quad \beta_{11} = \varphi_{11}(y_j)$$

$$\alpha_{1i} = \frac{-C_1}{B_1 + A_1\alpha_{1i-1}}; \quad \beta_{1i} = \frac{D_1 - A_1\beta_{1i-1}}{B_1 + A_1\alpha_{1i-1}}; \quad i = 2, \dots, n-1$$

$$u_{ij}^{s+1} = \alpha_{2i}u_{ij+1}^{s+1} + \beta_{2i}, \quad j = m-1, m-2, \dots, 2. \quad (11)$$

$$\alpha_{21} = 0; \quad \beta_{21} = \varphi_{13}(x_i)$$

$$\alpha_{2i} = \frac{-C_2}{B_2 + A_2\alpha_{2i-1}}; \quad \beta_{2i} = \frac{D_2 - A_2\beta_{2i-1}}{B_2 + A_2\alpha_{2i-1}}; \quad j = 2, \dots, m-1$$

Once the numerical approximations for the component u are fully computed, they are immediately incorporated into the subsequent difference scheme to explicitly evaluate the corresponding component v:

$$\tilde{v}_{ij} = \alpha_{3i}\tilde{v}_{i+1j} + \beta_{3i}, \quad i = n-1, n-2, \dots, 2. \quad (12)$$

$$\alpha_{31} = 0; \quad \beta_{31} = \varphi_{21}(y_j)$$

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$$\alpha_{3i} = \frac{-C_3}{B_3 + A_3\alpha_{3i-1}}; \quad \beta_{3i} = \frac{D_3 - A_3\beta_{3i-1}}{B_3 + A_3\alpha_{3i-1}}; \quad i = 2, \dots, n-1$$

$$v_{ij}^{s+1} = \alpha_{4i} v_{ij+1}^{s+1} + \beta_{4i}, \quad j = m-1, m-2, \dots, 2. \tag{13}$$

$$\alpha_{41} = 0; \quad \beta_{41} = \varphi_{23}(x_i)$$

$$\alpha_{4i} = \frac{-C_4}{B_4 + A_4\alpha_{4i-1}}; \quad \beta_{4i} = \frac{D_4 - A_4\beta_{4i-1}}{B_4 + A_4\alpha_{4i-1}}; \quad j = 2, \dots, m-1$$

By iteratively applying these recursive relations, the spatial distributions of both components are successfully resolved. The proposed fractional-step scheme sequentially sweeps along the orthogonal spatial axes, implicitly solving for the x -direction in the first fractional step and the y -direction in the subsequent step. In this study, the equations were solved using the backward sweep procedure. The initial sweep coefficients are strictly governed by the prescribed Dirichlet boundary conditions, enabling the rigorous recursive computation of the interior nodal values. Applying this method to system (1) leads to a reformulation of the problem in terms of the following non-stationary system of equations. To carry out this transformation, one may employ the implicit scheme with variable directions (the MAD method — *method of alternating directions*). In some sources, this approach is also referred to as the *method of fractional steps* [1]. A predefined tolerance criterion ε dictates the convergence of the numerical algorithm. The iterative pseudo-time stepping procedure is terminated once the maximum absolute deviation between two successive iterations falls below this threshold, thereby yielding the steady-state numerical solution:

$$\max\left(\max_{0\le i\le n, 0\le j\le m} |u_{ij}^{s+1} - u_{ij}^{s}|, \max_{0\le i\le n, 0\le j\le m} |v_{ij}^{s+1} - v_{ij}^{s}|\right) < \varepsilon \tag{14}$$

4 Results and Discussion

The proposed numerical algorithm and finite difference schemes were implemented to obtain the solution for the coupled system. Three distinct numerical experiments were conducted to evaluate the robustness and accuracy of the method under various conditions, including nonlinear and interconnected scenarios. In each experiment, the initial approximations for the functions u, v and

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tolerance ($\varepsilon = 10^{-3}$) were strategically selected to ensure convergence towards the steady-state solution and evaluate the overall accuracy of the proposed scheme.

Experiment 1: Validation with Exponential Smooth Solutions

$$\begin{cases} \frac{\partial}{\partial x}(v\frac{\partial}{\partial x}u) + \frac{\partial}{\partial y}(v\frac{\partial}{\partial y}u) = 0 \\ \frac{\partial}{\partial x}(u\frac{\partial}{\partial x}v) + \frac{\partial}{\partial y}(u\frac{\partial}{\partial y}v) = 0 \end{cases} \quad (15.1)$$

In the first case, the algorithm was tested using (15.1) test system; Exponential functions $u(x,y) = e^{x+y}$ and $v(x,y) = e^{-x-y}$ are Manufactured exact solutions. The results confirmed that the Method of Alternating Directions (MAD) effectively handles rapid spatial variations without losing precision. The functions on the boundary and initial points were chosen as follows:

$$\begin{cases} u|_{t=0} = e^{x+y} \\ v|_{t=0} = e^{-x-y} \end{cases} \begin{cases} u(0,y) = e^{y} \\ u(1,y) = e^{1+y} \\ u(x,0) = e^{x} \\ u(x,1) = e^{x+1} \end{cases} \begin{cases} v(0,y) = e^{-y} \\ v(1,y) = e^{-1-y} \\ v(x,0) = e^{-x} \\ v(x,1) = e^{-x-1} \end{cases} \quad (15.2)$$

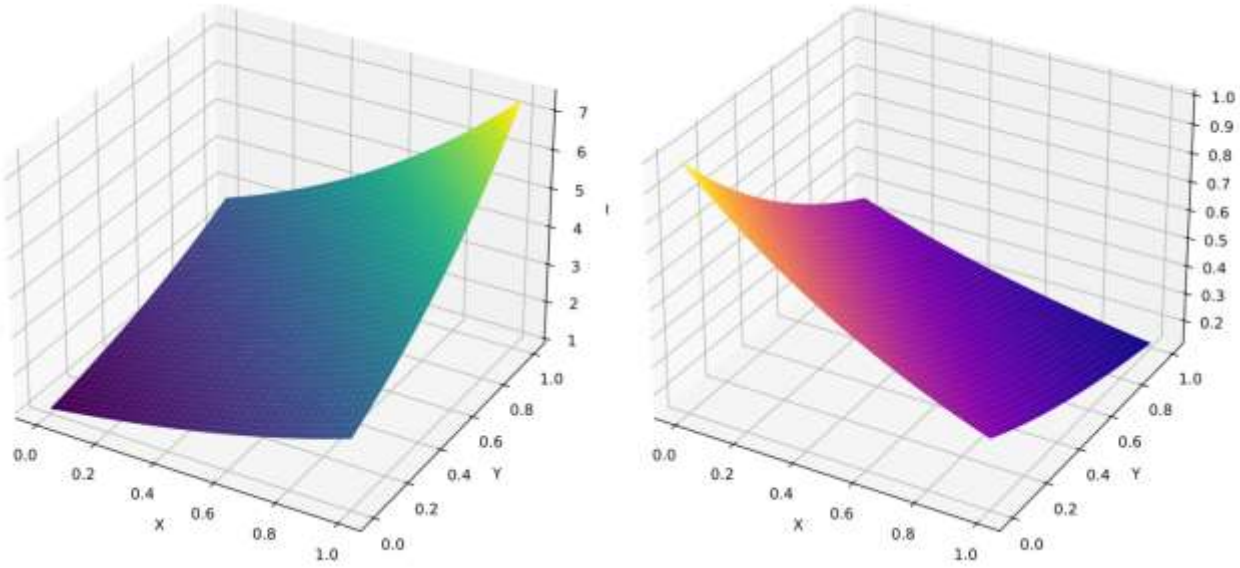

Figure 1: Left: Numerical distribution of the component $u(x,y)$ for Test 1, validating the MAD method against the manufactured exponential exact solution $v(x,y) = e^{x+y}$.

Right: Numerical distribution of the component $v(x,y)$, illustrating the stability of the algorithm for the exponential exact solution $v(x,y) = e^{-x-y}$.

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Experiment 2: Convergence Analysis for Periodic Boundary Conditions

$$\begin{cases} \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = 2\pi^2(1-u) \\ \frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v = 2\pi^2(1-v) \end{cases} \quad (16.1)$$

The second experiment (16.1) utilized trigonometric manufactured solutions to assess the scheme's performance under oscillating conditions:

$$\begin{cases} u(x,y) = \sin(\pi x)\sin(\pi y) + 1 \\ v(x,y) = \cos(\pi x)\cos(\pi y) + 1 \end{cases}$$

It was concluded that the Thomas algorithm maintains computational efficiency even as the grid is refined, providing a stable convergence path for the u and v variables. The functions on the boundary and initial points were chosen as follows:

$$\begin{cases} u|_{t=0} = 1 \\ v|_{t=0} = 1 \end{cases} \begin{cases} u(0,y) = 1 \\ u(1,y) = 1 \\ u(x,0) = 1 \\ u(x,1) = 1 \end{cases} \begin{cases} v(0,y) = \cos(\pi y) + 1 \\ v(1,y) = -\cos(\pi y) + 1 \\ v(x,0) = \cos(\pi x) + 1 \\ v(x,1) = -\cos(\pi x) + 1 \end{cases} \quad (16.2)$$

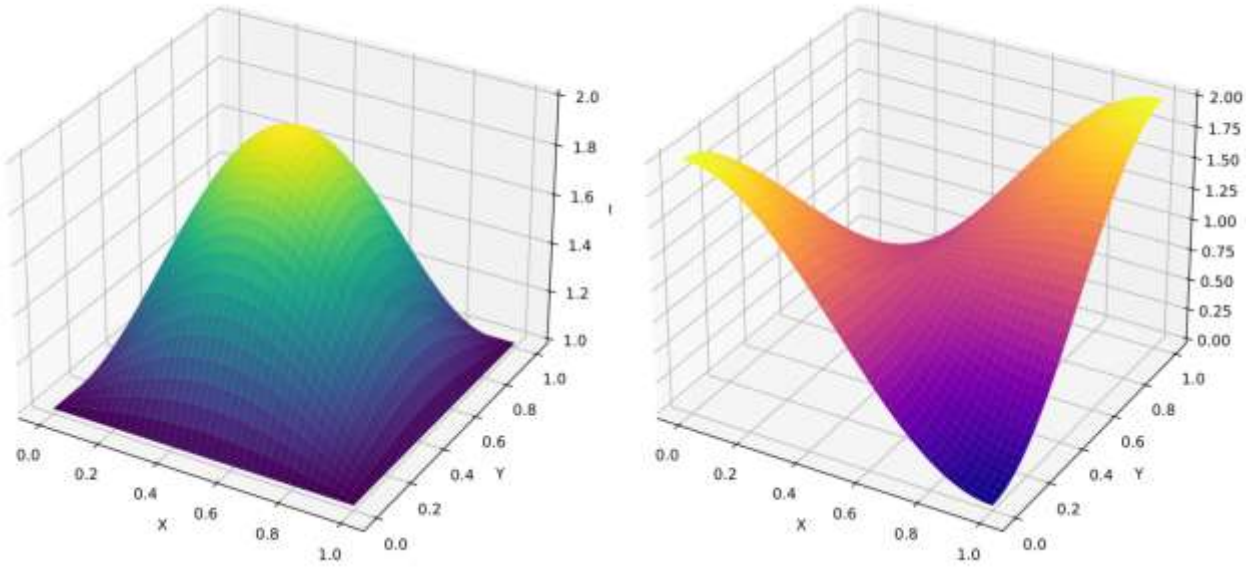

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Figure 2: Left: Numerical solution of the $u(x,y)$ component for Test 2, demonstrating the convergence of the Thomas algorithm for a stationary nonlinear boundary value problem.

Right: Numerical distribution of the component $v(x,y)$.

Experiment 3: Nonlinear Coupled System and Stress Testing

To evaluate the effectiveness of the algorithm on nonlinear and interconnected systems, a complex coupled problem was considered:

$$\begin{cases} \Delta u + u \cdot v = F_1(x,y) \\ \Delta v + u^2 = F_2(x,y) \end{cases} \quad (17.1)$$

System (17.1) is employed as a test model, given that its exact analytical solution can be explicitly constructed to verify the numerical accuracy:

$$\begin{cases} u(x,y) = \sin(\pi x)\sin(\pi y) + 1 \\ v(x,y) = e^{-x-y} + 1 \end{cases}$$

This final experiment demonstrated the robustness of the linearization technique and the overall ability of the solver to maintain high accuracy despite nonlinear interactions between the functions. The functions on the boundary and initial points were chosen as follows:

$$\begin{cases} u|_{t=0} = 1 \\ v|_{t=0} = 1 \end{cases} \begin{cases} u(0,y) = 1 \\ u(1,y) = 1 \\ u(x,0) = 1 \\ u(x,1) = 1 \end{cases} \begin{cases} v(0,y) = e^{-y} + 1 \\ v(1,y) = e^{-1-y} + 1 \\ v(x,0) = e^{-x} + 1 \\ v(x,1) = e^{-x-1} + 1 \end{cases} \quad (17.2)$$

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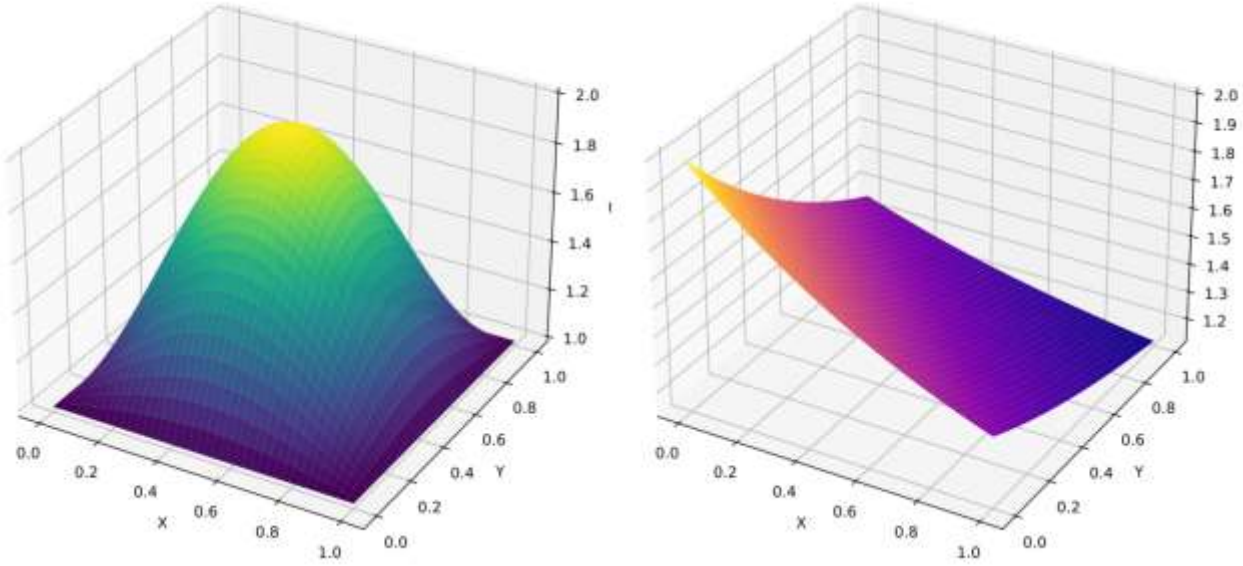

Figure 3: Numerical solutions for Test 3: (left) spatial distribution of the component $u(x, y)$ and (right) surface profile of the corresponding component $v(x, y)$, demonstrating the stability of the MAD method.

5 Conclusion

The present study successfully established a robust numerical framework grounded in the Method of Alternating Directions (MAD) and the Thomas algorithm for the resolution of complex nonlinear coupled systems. A comprehensive synthesis of the results obtained from three distinct numerical experiments reveals that the strategic selection of initial approximations serves as a fundamental catalyst for computational efficiency, ensuring stable convergence toward steady-state solutions across diverse physical scenarios. The present research serves as a comprehensive extension and advancement of the preliminary results established in the previous study conducted by Khojiev and Islomov [1]. While the earlier work introduced the fundamental numerical modeling of the nonlinear stationary heat conduction problem, it was primarily limited to Picard linearization and was exclusively validated using exponential test functions without extensive graphical visualization of the results.

In this work, significant enhancements have been implemented to address these limitations. First, the more advanced Newton linearization method was integrated alongside the Picard approach to achieve faster convergence rates for highly nonlinear interconnected systems. Second, the numerical framework was subjected to rigorous stress testing through three distinct experimental cases

involving diverse classes of analytical functions beyond simple exponentials. Finally, a complete qualitative and quantitative analysis was performed, including the generation of detailed graphical results that demonstrate the solver's stability and accuracy across refined grids.

Furthermore, systematic grid refinement investigations highlighted a direct correlation between the discretization parameter h and the precision of the numerical output, where the convergence toward the exact analytical solution confirmed the inherent accuracy of the second-order finite-difference approximations employed.

The validation provided by the third experiment was particularly significant, as it demonstrated the solver's exceptional reliability in managing nonlinear coupling between the variables u and v through advanced linearization techniques and the Method of Manufactured Solutions [13]. These findings collectively suggest that finite-difference-based methodologies offer a computationally stable and highly accurate pathway for simulating complex interconnected differential systems. Consequently, future endeavors will be directed toward optimizing these numerical schemes for even more intricate classes of nonlinear equations to enhance predictive modeling in computational science.

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