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SOME APPLICATIONS OF STOKES' FORMULA IN COMPLEX SPACE

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Abstract

This article discusses Stokes' formula in multidimensional complex spaces and some of its applications.

Key words

Stokes formula, simplex, directed surface, multidimensional space.

In this article, we'll look at the formula found by Sir George Stokes, known as Stokes' formula. For this, we present the following theorem:

Theorem. Let M be an oriented m-dimensional manifold, let σ be a p-dimensional chain, and let ω be a different form of degree $p-1$. If M, σ, $d\sigma$ and ω belong to class C^1, then the formula

$$\int\limits_{\sigma} d\omega = \int\limits_{\partial\sigma} \omega \tag{1}$$

holds ([1]).

This formula is the Stokes' formula.

From the definitions of chains and integrals adopted by us, it follows that the theorem it is sufficient to consider the case when the p-dimensional chain $S=\{P_0,P_1,...,P_p\}$ is a simplex. On $\overline{S}$, for each point $P\in\overline{S}$, we introduce coordinates $(t_1,t_2,...,t_p)$ given in the form of a linear combination of vectors, where

$$P = P_0 + \sum_{\nu=1}^{p} t_\nu (P_\nu - P_0) = \sum_{\nu=0} t_\nu P_\nu,\ t_0 = 1 - \sum_{\nu=1}^{p} t_\nu . \tag{2}$$

In these coordinates, the form ω can be written as a sum of p-forms. For example, we may take

$$\omega = f(t_1,...,t_p)dt_2 \wedge ... \wedge dt_p. \tag{3}$$

Let S_ν be the face of the simplex S opposite to the vertex P_ν. The equation of this face has the form $t_\nu = 0$ for $(\nu = 0,..., p)$. The orientations of these faces agree with the orientation of S. In particular, the face S_0 has the orientation of $\{P_1 \ldots P_p\}$, and the face S_1 has the orientation of $\{P_0, P_2 \ldots P_p\}$. Using form (3), we compute the integral over the oriented boundary ∂S of the simplex:

$$\int_{\partial S}\omega=\sum_{\nu=0}^{p}\int_{S_\nu}\omega=\int_{S_0}\omega+\int_{S_1}\omega \tag{4}$$

(On the remaining faces S_ν, $\nu \geq 2$, we take $\omega = 0$, since $t_\nu = 0$ and hence $dt_\nu = 0$.)

On the faces S_0 and S_1 we can use the coordinates $(t_2,\ldots,t_p) = t$. On the face S_0, the order of these coordinates agrees with the orientation. If t_1 is expressed as a (linear) function $\tau(t)$ of the remaining coordinates, then it is determined from the equation of the face $t_0 = 1 - \sum_{\nu=1}^{p} t_\nu = 0$.

From the definition of the integral taken over a form, we have the following:

$$\int_{S_0}\omega=\int_{S_1'} f[\tau(t),t]dt, \tag{5}$$

Here S_1' is a non-oriented simplex (the projection of S_0 into the space of variables, where $(t_2,\ldots,t_p) = t$), and $dt = dt_2 \ldots\ldots dt_p$ is the volume element.

We introduce the characteristic function $\chi(t)$ of the simplex S_1' (that is, the function equal to 1 on S_1' and 0 outside S_1'). Then we can rewrite the right-hand side of (5) as an integral over the whole space:

$$\int_{S_0}\omega=\int f(\tau,t)\chi(t)dt.$$

On S_1, the orientation order of the coordinates $(t_2,\ldots,t_n)$ is opposite to the direction, but $t_1 = 0$. Therefore we obtain:

$$\int_{S_1}\omega=-\int f(0,t)\chi(t)dt$$

and we prove (4).

$$\int_{\partial S}\omega=\int\{f(\tau,t)-f(0,t)\}\chi(t)dt, \tag{6}$$

here the integral is taken over the whole space.

On the other hand, $d\omega = \frac{\partial f}{\partial t_1} dt_1 \wedge dt_2 \wedge \ldots \wedge dt_p$,

$$\int_S d\omega = \int \chi(t)dt \int_0^{\tau(t)} \frac{\partial f}{\partial t_1} dt_1 = \int \chi(t)\{f(\tau,t) - f(0,t)\}dt.$$

This integral coincides with (6). ▷

Examples.

1. Let $p=1$ and m arbitrary. A one-dimensional chain σ consists of a curve, and its boundary $d\sigma$ consists of two points, with orientation: (a) negative and (b) positive. A 0-form ω is a function f. Stokes' formula reduces to the Newton–Leibniz formula:

$$\int_\sigma df = \int_{\partial\sigma} f = f(b) - f(a).$$

2. Let $m = p = 2$. The chain σ represents a two-dimensional region D.

A first-order differential form has the form

$$\omega = f_1 dt_1 + f_2 dt_2,$$

$$d\omega = \left(\frac{\partial f_2}{\partial t_1} - \frac{\partial f_1}{\partial t_2} \right) dt_1 \wedge dt_2.$$

In this case, Stokes' formula reduces to (that is, becomes) the Riemann–Green formula:

$$\iint_D \left(\frac{\partial f_2}{\partial t_1} - \frac{\partial f_1}{\partial t_2} \right) dt_1 dt_2 = \int_{\partial D} (f_1 dt_1 + f_2 dt_2).$$

3. Let $m=3,\ p=2;$ let the chain σ be a surface S in $\mathbb{R}^3$.

Formula (1) reduces to the classical Stokes formula:

$$\int \left(\frac{\partial f_2}{\partial t_1} - \frac{\partial f_1}{\partial t_2} \right) dt_1 \wedge dt_2 + \left(\frac{\partial f_3}{\partial t_1} - \frac{\partial f_1}{\partial t_3} \right) dt_1 \wedge dt_3 +$$

$$+ \left(\frac{\partial f_1}{\partial t_2} - \frac{\partial f_2}{\partial t_1} \right) dt_2 \wedge dt_1 = \int_{\partial S} f_1 dt_1 + f_2 dt_2 + f_3 dt_3.$$

(the remainder continues with the analogous term for the remaining pair of variables).

4. Let $m = p = 3$. The chain σ is a region D in $\mathbb{R}^3$.

Stokes' formula becomes the Ostrogradsky (Gauss divergence) formula:

$$\int_D \left(\frac{\partial f_{23}}{\partial t_1} - \frac{\partial f_{13}}{\partial t_2} + \frac{\partial f_{12}}{\partial t_3} \right) dt_1 \wedge dt_2 \wedge dt_3 = \int_{\partial D} f_{23} dt_2 \wedge dt_3 + f_{13} dt_1 \wedge dt_3 + f_{12} dt_1 \wedge dt_2$$

From Stokes' formula we obtain two important consequences:

1) The integral of a closed form $(d\omega = 0)$ over a null-homologous cycle ($\sigma = \partial \sum$) is zero:

$$\int_\sigma \omega = \int_{\partial \sum} \omega = \int_{\sum} d\omega = 0$$

2) The integral of an exact form $(\omega = d\omega_1)$ over any cycle $(\partial\sigma = 0)$ is zero:

$$\int_\sigma \omega = \int_\sigma d\omega_1 = \int_{\partial\sigma} \omega_1 = 0.$$

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SUMMARY:

This article discusses Stokes' formula in multidimensional spaces and some of its applications.